MATH2068 MATHEMATICAL ANALYSIS II (2021-22)

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1. Differentiation

Throughout this section, let I be an open interval (not necessarily bounded) and let f be a real-valued function defined on I.

Definition 1.1. *Let $c \in I$. We say that f is differentiable at c if the following limit exists:*

$$\lim_{x\to c}\frac{f(x)-f(c)}{x-c}.$$

In this case, we write $f'(c)$ for the above limit and we call it the derivative of f at c. We say that if f is differentiable on I if $f'(x)$ exists for every point x in I.

Proposition 1.2. *Let $c \in I$. Then $f'(c)$ exists if and only if there is a function φ defined on I such that the function φ is continuous at c and*

$$f(x)-f(c)=\varphi(x)(x-c)$$

for all $x \in I$.
In this case, $\varphi(c)=f'(c)$.

Proof. Assume that $f'(c)$ exists. Define a function $\varphi : I \to \mathbb{R}$ by

$$\varphi(x)=\begin{cases}\frac{f(x)-f(c)}{x-c} & \text{if } x\neq c;\\ f'(c) & \text{if } x=c.\end{cases}$$

Clearly, we have $f(x)-f(c)=\varphi(x)(x-c)$ for all $x \in I$. We want to show that the function φ is continuous at c. In fact, let $\varepsilon>0$, by the definition of the limit of a function, there is $\delta>0$ such that

$$|f'(c)-\frac{f(x)-f(c)}{x-c}|<\varepsilon$$

whenever $x\in I$ with $0<|x-c|<\delta$. Therefore, we have $|f'(c)-\varphi(x)|<\varepsilon$ as $x\in I$ with $0<|x-c|<\delta$. Since $\varphi(c)=f'(c)$, we have $|f'(c)-\varphi(x)|<\varepsilon$ as $x\in I$ with $|x-c|<\delta$, hence the function φ is continuous at c as desired.
The converse is clear since $\varphi(x)=\frac{f(x)-f(c)}{x-c}$ if $x\neq c$. The proof is complete. □

Proposition 1.3. *Using the notation as above, if f is differentiable at c, then f is continuous at c.*

Proof. By using Proposition 1.2, if $f'(c)$ exists, then there is a function φ defined on I such that the function φ is continuous at c and we have $f(x)-f(c)=\varphi(x)(x-c)$ for all $x\in I$. This implies that $\lim_{x\to c} f(x)=f(c)$, so f is continuous at c as desired. □

Remark 1.4. *In general, the converse of Proposition 1.3 does not hold, for example, the function $f(x):=|x|$ is a continuous function on $\mathbb{R}$ but $f'(0)$ does not exist.*

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Proposition 1.5. *Let f and g be the functions defined on I. Assume that f and g both are differentiable at $c \in I$. We have the following assertions.*

(i) $(f+g)'(c)$ *exists and* $(f+g)'(c) = f'(c) + g'(c)$.

(ii) The product $(f \cdot g)'(c)$ *exists and* $(f \cdot g)'(c) = f'(c)g(c) + f(c)g'(c)$.

(iii) If $g(c) \neq 0$, *then we have* $(\frac{f}{g})'(c)$ *exists and* $(\frac{f}{g})'(c) = \frac{f'(c)g(c) - f(c)g'(c)}{g(c)^2}$.

Proof. Part (i) clearly follows from the definition of the limit of a function.
For showing Part (ii), note that we have

$$\frac{f(x)g(x) - f(c)g(c)}{x-c} = \frac{f(x)-f(c)}{x-c} g(x) + f(c) \frac{g(x)-g(c)}{x-c}$$

for all $x \in I$ with $x \neq c$. From this, together with Proposition 1.3, Part (ii) follows.
For Part (iii), by using Part (ii), it suffices to show that $(\frac{1}{g})'(c) = -\frac{g'(c)}{g(c)^2}$. In fact, $g'(c)$ exists, so g is continuous at c. Since $g(c) \neq 0$, there is $\delta_1 > 0$ so that $g(x) \neq 0$ for all $x \in I$ with $|x-c| < \delta_1$. Then we have

$$\frac{1}{x-c}\left(\frac{1}{g(x)} - \frac{1}{g(c)}\right) = \frac{1}{x-c}\left(\frac{g(c)-g(x)}{g(x)g(c)}\right)$$

for all $x \in I$ with $0 < |x-c| < \delta_1$. By taking $x \to c$, we see that $(\frac{1}{g})'(c)$ exists and $(\frac{1}{g})'(c) = \frac{-g'(c)}{g(c)^2}$. The proof is complete. □

Proposition 1.6. (Chain Rule): *Let f, g be functions defined on $\mathbb{R}$. Let $d = f(c)$ for some $c \in \mathbb{R}$. Suppose that $f'(c)$ and $g'(d)$ exist. Then the derivative of composition $(g \circ f)'(c)$ exists and $(g \circ f)'(c) = g'(d)f'(c)$.*

Proof. By using Proposition 1.2, we want to find a function $\varphi : \mathbb{R} \to \mathbb{R}$ such that

$$g \circ f(x) - g \circ f(c) = \varphi(x)(x-c)$$

for all $x \in \mathbb{R}$ and the function $\varphi(x)$ is continuous at c, and so $(g \circ f)'(c) = \varphi(c)$.
Let $y = f(x)$. By using Proposition 1.2 again, there is a function and $\beta(y)$ so that $g(y) - g(d) = \beta(y)(y-d)$ for all $y \in \mathbb{R}$ and $\beta(y)$ is continuous at d. Similarly, there is a function $\alpha(x)$ we have $f(x) - f(c) = \alpha(x)(x-c)$ for all $x \in \mathbb{R}$ and $\alpha(x)$ is continuous at c. These two equations imply that

$$g \circ f(x) - g \circ f(c) = \beta(f(x))(f(x) - f(c)) = \beta(f(x))\alpha(x)(x-c)$$

for all $x \in \mathbb{R}$. Let $\varphi(x) := \beta(f(x)) \cdot \alpha(x)$ for $x \in \mathbb{R}$. Since $\beta(d) = g'(d)$ and $\alpha(c) = f'(c)$, we see that $\varphi(c) = \beta(f(c))\alpha(c) = g'(d)f'(c)$. It remains to show that the function φ is continuous at c. In fact, $f'(c)$ exists, so f is continuous at c, and hence the composition $\beta \circ f(x)$ is continuous at c. In addition, the function α is continuous at c. Therefore, the function $\varphi := (\beta \circ f) \cdot \alpha$ is continuous at c, and so $(g \circ f)'(c)$ exists with $(g \circ f)'(c) = \varphi(c) = g'(d)f'(c)$. The proof is complete. □

Proposition 1.7. *Let I and J be open intervals. Let f be a strictly increasing function from I onto J. Let $d = f(c)$ for $c \in I$. Assume that $f'(c)$ exists and the inverse of f, write $g := f^{-1}$, is continuous at d. If $f'(c) \neq 0$, then $g'(d)$ exists and $g'(d) = \frac{1}{f'(c)}$.*

Proof. Let $y = f(x)$. Note that by using Proposition 1.2, there is a function F on I such that $f(x) - f(c) = F(x)(x-c)$ for all $x \in I$ and F is continuous at c with $F(c) = f'(c) \neq 0$. F is continuous at c, so there are open intervals I_1 and J_1 such that $c \in I_1 \subseteq I$ and $d \in f(I_1) = J_1$, moreover, $F(x) \neq 0$ for all $x \in I_1$. Note that since $f(x) - f(c) = F(x)(x-c)$, we have $y - d = f(g(y)) - f(g(c)) = F(g(y))(g(y) - g(d))$ for all $y \in J_1$. Since $F(x) \neq 0$ for all $x \in I_1$, we have $g(y) - g(d) = F(g(y))^{-1}(y-d)$ for all $y \in J_1$. Note that the function $F(g(y))^{-1}$ is continuous at d. Thus, $g'(d)$ exists and $g'(d) = F(g(d))^{-1} = \frac{1}{f'(c)}$ as desired. □

Definition 1.8. *Let D be a non-empty subset of $\mathbb{R}$ and let g be a real-valued function defined on D.*

(i) We say that g has an absolute maximum (resp. absolute minimum) at a point $c \in D$ if $g(c) \geq g(x)$ (resp. $g(c) \leq g(x)$) for all $x \in D$.
In this case, c is called an absolute extreme point of g.

(ii) We say that g has a local maximum (resp. local minimum) at a point $c \in D$ if there is $r > 0$ such that $(c-r, c+r) \subseteq D$ and $g(c) \geq g(x)$ (resp. $g(c) \leq g(x)$) for all $x \in (c-r, c+r)$.
In this case, c is called a local extreme point of g.

Remark 1.9. *Note that an absolute extreme point of a function g need not be a local extreme point, for example if $g(x) := x$ for $x \in [0,1]$, then g has an absolute maximum point at $x = 1$ of g but 1 is not a local maximum point of g.*

Proposition 1.10. *Let I be an open interval and let f be a function on I. Assume that f has a local extreme point at $c \in I$ and $f'(c)$ exists. Then $f'(c) = 0$.*

Proof. Without lost the generality, we may assume that f has local minimum at c. Then there is $r > 0$ such that $f(x) \geq f(c)$ for $x \in (c-r, c+r) \subseteq I$. Since $f'(c)$ exists, by using Proposition 1.2, there is a function φ defined on I such that $f(x) - f(c) = \varphi(x)(x-c)$ for all $x \in I$ and φ is continuous at c with $\varphi(c) = f'(c)$. Thus, we have $\varphi(x)(x-c) \geq 0$ for all $x \in (c-r, c+r)$. From this we see that $\varphi(x) \geq 0$ as $x \in (c, c+r)$, similarly, $\varphi(x) \leq 0$ as $x \in (c-r, c)$. The function φ is continuous at c, so $\varphi(c) = 0$ and hence $f'(c) = \varphi(c) = 0$ as desired. □

Proposition 1.11. Rolle's Theorem*: Let $f : [a,b] \to \mathbb{R}$ be a continuous function. Assume that $f'(x)$ exists for all $x \in (a,b)$ and $f(a) = f(b)$. Then there is a point $c \in (a,b)$ such that $f'(c) = 0$.*

Proof. Recall a fact that every continuous function defined a compact attains absolute points, that is, there are c_1 and c_2 such that $f(c_1) = \min_{x \in [a,b]} f(x)$ and $f(c_2) = \max_{x \in [a,b]} f(x)$, hence, $f(c_1) \leq f(x) \leq f(c_2)$ for all $x \in [a,b]$. If $f(c_1) = f(c_2)$, then $f(x) \equiv f(c_1) = f(c_2)$ for all $x \in [a,b]$, so $f'(x) \equiv 0$ for all $x \in (a,b)$.
Otherwise, suppose that $f(c_1) < f(c_2)$. Since $f(a) = f(b)$, we have $c_1 \in (a,b)$ or $c_2 \in (a,b)$. We may assume that $c_1 \in (a,b)$. Then $x = c_1$ is a local minimum point of f. Therefore, $f'(c_1) = 0$ by using Proposition 1.10. □

Theorem 1.12. Main Value Theorem: *If $f : [a,b] \to \mathbb{R}$ is a continuous function and is differentiable on (a,b), then there is a point $c \in (a,b)$ such that $f(b) - f(a) = f'(c)(b-a)$.*

Proof. Define a function $\varphi : [a,b] \to \mathbb{R}$ by

$$\varphi(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b-a}(x-a)$$

for $x \in [a,b]$. Note that the function φ is continuous on $[a,b]$ with $\varphi(a) = \varphi(b) = 0$, in addition, $\varphi'(x)$ exists for all $x \in (a,b)$. The Rolle's Theorem implies that there is a point $c \in (a,b)$ such that

$$0 = \varphi'(c) = f'(c) - \frac{f(b) - f(a)}{b-a}.$$

The proof is complete. □

Corollary 1.13. *Assume that $f : [a,b] \to \mathbb{R}$ is a continuous function and is differentiable on (a,b). If $f' \equiv 0$ on (a,b), then f is a constant function.*

Proof. Fix any point $z \in (a,b)$. Let $x \in (z,b]$. By using the Mean Value Theorem, there is a point $c \in (z,x)$ such that $f(x) - f(z) = f'(c)(x-z)$. If $f' \equiv 0$ on (a,b), so $f(x) = f(z)$ for all $x \in [z,b]$. Similarly, we have $f(x) = f(z)$ for all $x \in [a,z]$. The proof is complete. □

Definition 1.14. *We call a function f is a C^1-function on I if $f'(x)$ exists and continuous on I. In addition, we define the n-derivatives of f by $f^{(n)}(x) := f^{(n-1)}(x)$ for $n \geq 2$, provided it exists. In this case, we say that f is a C^n-function on I. In particular, we call f a C^∞-function (or smooth function) if f is a C^n-function for all $n = 1, 2....$*
For example, the exponential function $\exp x$ is a very important example of smooth function on $\mathbb{R}$.

Corollary 1.15. Inverse Mapping Theorem: *Let f be a C^1-function on an open interval I and let $c \in I$. Assume that $f'(c) \neq 0$. Then there is $r > 0$ such that the function f is a strictly monotone function on $(c-r, c+r) \subseteq I$. If we let $J := f(c-r, c+r))$, then the inverse function $g := f^{-1} : J \to (c-r, c+r)$ is also a C^1-function.*

Proof. We may assume that $f'(c) > 0$. $f'(x)$ is continuous on I, so there is $r > 0$ such that $f'(x) > 0$ for all $x \in (c-r, c+r) \subseteq I$. For any x_1 and x_2 in $(c-r,, c+r)$ with $x_1 < x_2$, by using the Mean Value Theorem, we have $f(x_2) - f(x_1) = f'(v)(x_2 - x_1)$ for some $v \in (x_1, x_2)$, and hence $f(x_2) > f(x_1)$. Therefore the restriction of f on $(c-r, c+r)$ is a strictly increasing function, thus, it is an injection. Let $J := f((c-r, c+r))$. Then J is an interval by the Immediate Value Theorem. Moreover, J is an open interval because f is strictly increasing. Also, if we let $g = f^{-1}$ on J, then g is continuous on J due to the fact that every continuous bijection on a compact set is a homeomorphism. Therefore, by Proposition 1.7, we see that $g'(y)$ exists on J and $g'(y) = \frac{1}{f'(x)}$ for $y = f(x)$ and $x \in (c-r, c+r)$. Therefore, g is a C^1 function on J. The proof is complete. □

Proposition 1.16. Cauchy Mean Value Theorem: *Let $f, g : [a,b] \to \mathbb{R}$ be continuous functions with $g(a) \neq g(b)$. Assume that f, g are differentiable functions on (a,b) and $g'(x) \neq 0$ for all $x \in (a,b)$. Then there is a point $c \in (a,b)$ such that $\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$.*

Proof. Define a function ψ on $[a,b]$ by $\psi(x) = f(x) - f(a) - \frac{f(b)-f(a)}{g(b)-g(a)}(g(x) - g(a))$ for $x \in [a,b]$. Then by using the similar argument as in the Mean Value Theorem, the result follows. □

Theorem 1.17. Lagrange Remainder Theorem: *Let f be a $C^{(n+1)}$ function defined on (a,b). Let $x_0 \in (a,b)$. Then for each $x \in (a,b)$, there is a point c between x_0 and x such that*

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k + \frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1}.$$

Proof. We may assume that $x_0 < x < b$. **Case:** We first assume that $f^{(k)}(x_0) = 0$ for all $k = 0, 1, ..., n$. Put $g(t) = (t-x_0)^{n+1}$ for $t \in [x_0, x]$. Then $g'(t) = (n+1)(t-x_0)^n$ and $g(x_0) = 0$. Then by the Cauchy Mean Value Theorem, there is $x_1 \in (x_0, x)$ such that $\frac{f(x)}{g(x)} = \frac{f(x)-f(x_0)}{g(x)-g(x_0)} = \frac{f'(x_1)}{g'(x_1)}$. Using the same step for f' and g' on $[x_0, x_1]$, there is $x_2 \in (x_0, x_1)$ such that $\frac{f'(x_1)}{g'(x_1)} = \frac{f'(x_1)-f'(x_0)}{g'(x_1)-g'(x_0)} = \frac{f^{(2)}(x_2)}{g(2)(x_2)}$. To repeat the same step, there are $x_1, x_2, ..., x_{n+1}$ in (a,b) such that $x_k \in (x_0, x_{k-1})$ for $k = 1, 2, ..., n+1$ and

$$\frac{f(x)}{g(x)} = \frac{f'(x_1)}{g'(x_1)} = \cdots = \frac{f^{(n+1)}(x_{n+1})}{g^{(n+1)}(x_{n+1})}.$$

In addition, note that $g^{n+1}(x_{n+1}) = (n+1)!$. Therefore, we have $\frac{f(x)}{g(x)} = \frac{f^{(n+1)}(x_{n+1})}{(n+1)!}$, and hence $f(x) = \frac{f^{(n+1)}(x_{n+1})}{(n+1)!}(x-x_0)^{n+1}$. Note $x_{n+1} \in (x_0, x)$ and thus, the result holds for this case.

For the general case, put $G(x) = f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$ for $x \in (a,b)$. Note that we have $G(x_0) = G'(x_0) = \cdots = G^{(n)}(x_0) = 0$. Then by the Claim above, there is a point $c \in (x_0, x)$ such that $G(x) = \frac{G^{(n+1)}(c)}{(n+1)!}$. Since $G^{(n+1)}(c) = f^{(n+1)}(c)$, $f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k + \frac{f^{(n+1)}(c)}{(n+1)!}$. The proof is complete. □

Example 1.18. Recall that the exponential function e^x is defined by

$$e^x := \sum_{k=0}^{\infty} \frac{x^k}{k!} := \lim_{n\to\infty} \sum_{k=0}^{n} \frac{x^k}{k!}$$

for $x \in \mathbb{R}$. Note that the above limit always exists for all $x \in \mathbb{R}$ (shown in the last chapter).
Show that the natural base e is an irrational number.
Put $f(x) := e^x$ for $x \in \mathbb{R}$. It is a known fact f is a C^∞ function and $f^{(n)}(x) = e^x$ for all $x \in \mathbb{R}$. Fix any $x > 0$. Then by the Lagrange Theorem, for each positive integer n, there is $c_n \in (0, x)$ such that

$$f(x) = \sum_{k=0}^{n} \frac{x^k}{k!} + \frac{e^{c_n}}{(n+1)!} x^{n+1}.$$

In particular, taking $x = 1$, we have

$$0 < \frac{e^{c_n}}{(n+1)!} = e - \sum_{k=0}^{n} \frac{1}{k!} < \frac{3}{(n+1)!}$$

for all positive integer n. Now if $e = p/q$ for some positive integers p and q, and thus, we have

$$0 < \frac{p}{q} - \sum_{k=0}^{n} \frac{1}{k!} < \frac{3}{(n+1)!}$$

for all $n = 1, 2...$ Now we can choose n large enough such that $(n!)\frac{p}{q} \in \mathbb{N}$. It leads to a contradiction because we have

$$0 < (n!)\frac{p}{q} - (n!)\sum_{k=0}^{n} \frac{1}{k!} < \frac{3(n!)}{(n+1)!} = \frac{3}{n+1} < 1.$$

Therefore, e is irrational.

Proposition 1.19. *Let f be a C^2 function on an open interval I and $x_0 \in I$. Assume that $f'(x_0) = 0$. Then f has local maximum (resp. local minimum) at x_0 if $f^{(2)}(x_0) < 0$ (resp. $f^{(2)}(x_0) > 0$).*

Proof. We assume that $f^{(2)}(x_0) > 0$. We want to show that x_0 is a local minimum point of f. The proof of another case is similar. Note that for any $x \in I \setminus \{x_0\}$. Then by the Lagrange Theorem, there is a point c between x_0 and x such that

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{1}{2} f^{(2)}(x_0)(x-x_0)^2 = f(x_0) + \frac{1}{2} f^{(2)}(x_0)(x-x_0)^2.$$

$f^{(2)}$ is continuous at x_0 and $f^{(2)}(x_0) > 0$, and so there is $r > 0$ such that $f^{(2)}(x) > 0$ for all $x \in (x_0 - r, x_0 + r) \subseteq I$. Therefore, we have

$$f(x) = f(x_0) + \frac{1}{2} f^{(2)}(x)(x-x_0)^2 \geq f(x_0)$$

for all $x \in (x_0 - r, x_0 + r)$ and thus, x_0 is a local minimum point of f as desired. □

Proposition 1.20. L'Hospital's Rule: *Let f and g be the differentiable functions on (a,b) and let $c \in (a,b)$ Assume that $f(c)=g(c)=0$, in addition, $g'(x)\neq 0$ and $g(x)\neq 0$ for all $x\in(a,b)\setminus\{c\}$. If the limit $L := \lim\limits_{x\to c}\dfrac{f'(x)}{g'(x)}$ exists, then so does $\lim\limits_{x\to c}\dfrac{f(x)}{g(x)}$, moreover, we have $L=\lim\limits_{x\to c}\dfrac{f(x)}{g(x)}$.*

Proof. Fix $c<x<b$. Then by the Cauchy Mean Value Theorem, there is a point $x_1\in(c,x)$ such that

$$\frac{f(x)}{g(x)}=\frac{f(x)-f(c)}{g(x)-g(c)}=\frac{f'(x_1)}{g'(x_1)}$$

$x_1\in(c,x)$, so if $L := \lim\limits_{x\to c}\dfrac{f'(x)}{g'(x)}$ exists, then $\lim\limits_{x\to c+}\dfrac{f(x)}{g(x)}$ exists and is equal to L.

Similarly, we also have $\lim\limits_{x\to c-}\dfrac{f(x)}{g(x)}=L$. The proof is finished. □

Proposition 1.21. *Let f be a function on (a,b) and let $c\in(a,b)$.*

(i) If $f'(c)$ exists, then the following limit exists (also called the symmetric derivatives of f at c):

$$f'(c)=\lim_{t\to 0}\frac{f(c+t)-f(c-t)}{2t}.$$

(ii) If $f^{(2)}(c)$ exists, then

$$f^{(2)}(c)=\lim_{t\to 0}\frac{f(c+t)-2f(c)+f(c-t)}{t^2}.$$

Proof. For showing (i), note that we have

$$f'(c)=\lim_{t\to 0+}\frac{f(c+t)-f(c)}{t}=\lim_{t\to 0-}\frac{f(c+t)-f(c)}{t}.$$

Putting $t=-s$ into the second equality above, we see that

$$f'(c)=\lim_{s\to 0+}\frac{f(c-s)-f(c)}{-s}.$$

To sum up the two equations above, we have

$$f'(c)=\lim_{t\to 0+}\frac{f(c+t)-f(c-t)}{2t}.$$

Similarly, we have $f'(c)=\lim\limits_{t\to 0-}\dfrac{f(c+t)-f(c-t)}{2t}$. Part (i) follows.

For showing Part (ii), let $h(t) := f(c+t)-2f(c)+f(c-t)$ for $t\in\mathbb{R}$. Then $h(0)=0$ and $h'(t)=f'(c+t)-f'(c-t)$. By using the L'Hospital's Rule and Part (i), we have

$$\lim_{t\to 0}\frac{f(c+t)-2f(c)+f(c-t)}{t^2}=\lim_{t\to 0}\frac{h'(t)}{(t^2)'}=\lim_{t\to 0}\frac{f'(c+t)-f'(c-t)}{2t}=f^{(2)}(c).$$

The proof is complete. □

Definition 1.22. *A function f defined on (a,b) is said to be convex if for any pair $a<x_1<x_2<b$, we have*

$$f((1-t)x_1+tx_2)\leq(1-t)f(x_1)+tf(x_2)$$

for all $t\in[0,1]$.

Proposition 1.23. *Let f be a C^2 function on (a,b). Then f is a convex function if and only if $f^{(2)}(x)\geq 0$ for all $x\in(a,b)$.*

Proof. For showing ($\Rightarrow$): assume that f is a convex function. Fix a point $c \in (a,b)$. f is convex, so we have $f(c) = f(\frac{1}{2}(c+t) + \frac{1}{2}(c-t)) \leq \frac{1}{2}f(c+t) + \frac{1}{2}f(c-t)$ for all $t \in \mathbb{R}$ with $c \pm t \in (a,b)$. By Proposition 1.21, we have

$$f^{(2)}(c) = \lim_{t \to 0} \frac{f(c+t) - 2f(c) + f(c-t)}{t^2}.$$

Therefore, we have $f^{(2)}(c) \geq 0$.
For ($\Leftarrow$), assume that $f^{(2)}(x) \geq 0$ for all $x \in (a,b)$. Fix $a < x_1 < x_2 < b$ and $t \in [0,1]$. Let $c := (1-t)x_1 + tx_2$. Then by the Lagrange Reminder Theorem, there are points $z_1 \in (x_1, c)$ and $z_2 \in (c, x_2)$ such that

$$f(x_2) = f(c) + f'(c)(x_2 - c) + \frac{1}{2}f^{(2)}(z_2)(x_2 - c)^2$$

and

$$f(x_1) = f(c) + f'(c)(x_1 - c) + \frac{1}{2}f^{(2)}(z_1)(x_1 - c)^2.$$

These two equations implies that

$$(1-t)f(x_1) + tf(x_2) = f(c) + (1-t)\frac{1}{2}f^{(2)}(z_1)(x_1 - c)^2 + t\frac{1}{2}f^{(2)}(z_2)(x_2 - c)^2 \geq f(c).$$

since $f^{(2)}(z_1)$ and $f^{(2)}(z_2)$ both are non-negative. Thus, f is convex. □

Corollary 1.24. *Let $p > 0$. The function $f(x) := x^p$ is convex on $(0, \infty)$ if and only if $p \geq 1$.*

Proof. Note that $f^{(2)}(x) = p(p-1)x^{p-2}$ for all $x > 0$. Then the result follows immediately from Proposition 1.23. □

Proposition 1.25. Netwon's Method: *Let f be a continuous real-valued function defined on $[a,b]$ with $f(a) < 0 < f(b)$ and $f(z) = 0$ for some $z \in (a,b)$. Assume that f is a C^2 function on (a,b) and $f'(x) \neq 0$ for all $x \in (a,b)$. Then there is $\delta > 0$ with $J := [z-\delta, z+\delta] \subseteq [a,b]$ which have the following property:*
if we fix any $x_1 \in J$ and let

$$x_{n+1} := x_n - \frac{f(x_n)}{f'(x_n)} \tag{1.1}$$

for $n = 1, 2, ...$, then we have $z = \lim x_n$.

Proof. We first choose $r > 0$ such that $[z-r, z+r] \subseteq (a,b)$. We fix any point $x_1 \in (z-r, z+r)$ with $x_1 \neq z$. Then by the Lagrange Remainder Theorem, there is a point ξ between z and x_1 such that

$$0 = f(z) = f(x_1) + f'(x_1)(z - x_1) + \frac{1}{2}f^{(2)}(\xi)(z - x_1)^2.$$

This, together with Eq 1.1 above, we have

$$x_2 - x_1 = -\frac{f(x_1)}{f'(x_1)} = z - x_1 + \frac{f^{(2)}(\xi)}{2f'(x_1)}(z - x_1)^2.$$

Therefore, we have

$$x_2 - z = \frac{f^{(2)}(\xi)}{2f'(x_1)}(z - x_1)^2. \tag{1.2}$$

Note that the functions $f'(x)$ and $f^{(2)}(x)$ are continuous on $[z-r,z+r]$ and $f'(x)\neq 0$, hence, there is $M>0$ such that $|\frac{f^{2)}(u)}{2f'(v)}|\leq M$ for all $u,v\in[z-r,z+r]$. Then the Eq 1.2 implies that

$$|x_2-z|=|\frac{f^{(2)}(\xi)}{2f'(x_1)}(z-x_1)^2|\leq M(z-x_1)^2. \tag{1.3}$$

Choose $\delta>0$ such that $M\delta<1$ and $J:=[z-\delta,z+\delta]\subseteq(z-r,z+r)$. Note that Now we take any $x_1\in J$. Eq 1.3 implies that $|x_2-z|\leq M\cdot|z-x_1|^2\leq(M\delta)\cdot|x_1-z|<\delta$. By using Eq 1.1 inductively, we have a sequence (x_n) in J such that

$$|x_{n+1}-z|\leq M\cdot|z-x_n|^2\leq(M\delta)\cdot|x_n-z|$$

for all $n=1,2....$ Therefore, we have

$$|x_{n+1}-z|\leq(M\delta)^n\cdot|x_1-z|$$

for all $n=1,2...$, thus, $\lim x_n=z$. The proof is complete. □

2. Riemann Integrable Functions

We will use the following notation throughout this chapter.

(i): All functions $f, g, h...$ are bounded real valued functions defined on $[a, b]$ and $m \leq f \leq M$ on $[a, b]$.

(ii): Let $P : a = x_0 < x_1 < .... < x_n = b$ denote a partition on $[a, b]$; Put $\Delta x_i = x_i - x_{i-1}$ and $\|P\| = \max \Delta x_i$.

(iii): $M_i(f, P) := \sup\{f(x) : x \in [x_{i-1}, x_i\}$; $m_i(f, P) := \inf\{f(x) : x \in [x_{i-1}, x_i\}$.
Set $\omega_i(f, P) = M_i(f, P) - m_i(f, P)$.

(iv): (the *upper sum* of f): $U(f, P) := \sum M_i(f, P)\Delta x_i$
(the *lower sum* of f). $L(f, P) := \sum m_i(f, P)\Delta x_i$.

Remark 2.1. *It is clear that for any partition on $[a, b]$, we always have*
(i) $m(b - a) \leq L(f, P) \leq U(f, P) \leq M(b - a)$.
(ii) $L(-f, P) = -U(f, P)$ *and* $U(-f, P) = -L(f, P)$.

The following lemma is the critical step in this section.

Lemma 2.2. *Let P and Q be the partitions on $[a, b]$. We have the following assertions.*
(i) If $P \subseteq Q$, then $L(f, P) \leq L(f, Q) \leq U(f, Q) \leq U(f, P)$.
(ii) We always have $L(f, P) \leq U(f, Q)$.

Proof. For Part (i), we first claim that $L(f, P) \leq L(f, Q)$ if $P \subseteq Q$. By using the induction on $l := \#Q - \#P$, it suffices to show that $L(f, P) \leq L(f, Q)$ as $l = 1$. Let $P : a = x_0 < x_1 < \cdots < x_n = b$ and $Q = P \cup \{c\}$. Then $c \in (x_{s-1}, x_s)$ for some s. Notice that we have

$$m_s(f, P) \leq \min\{m_s(f, Q), m_{s+1}(f, Q)\}.$$

So, we have

$$m_s(f, P)(x_s - x_{s-1}) \leq m_s(f, Q)(c - x_{s-1}) + m_{s+1}(f, Q)(x_s - c).$$

This gives the following inequality as desired.

$$L(f, Q) - L(f, P) = m_s(f, Q)(c - x_{s-1}) + m_{s+1}(f, Q)(x_s - c) - m_s(f, P)(x_s - x_{s-1}) \geq 0. \tag{2.1}$$

Now by considering $-f$ in the Inequality 2.1 above, we see that $U(f, Q) \leq U(f, P)$.
For Part (ii), let P and Q be any pair of partitions on $[a, b]$. Notice that $P \cup Q$ is also a partition on $[a, b]$ with $P \subseteq P \cup Q$ and $Q \subseteq P \cup Q$. So, Part (i) implies that

$$L(f, P) \leq L(f, P \cup Q) \leq U(f, P \cup Q) \leq U(f, Q).$$

The proof is complete. □

The following notion plays an important role in this chapter.

Definition 2.3. *Let f be a bounded function on $[a, b]$. The upper integral (resp. lower integral) of f over $[a, b]$, write $\overline{\int_a^b} f$ (resp. $\underline{\int_a^b} f$), is defined by*

$$\overline{\int_a^b} f = \inf\{U(f, P) : P \textit{ is a partation on } [a, b]\}.$$

(resp.

$$\underline{\int_a^b} f = \sup\{L(f,P) : P \textit{ is a partation on } [a,b]\}.)$$

Notice that the upper integral and lower integral of f must exist by Remark 2.1.

Remark 2.4. **Appendix**: We call a partially set $(I, \leq)$ a *directed set* if for each pair of elements i_1 and i_2 in I, there is $i_3 \in I$ such that $i_1 \leq i_3$ and $i_2 \leq i_3$.
A *net* in $\mathbb{R}$ is a real-valued function f defined on a directed set I, write $f = (x_i)_{i \in I}$, where $x_i := f(i)$ for $i \in I$.
We say that a net (x_i) converges to a point $L \in \mathbb{R}$ (call a limit of (x_i)) if for any $\varepsilon > 0$, there is $i_0 \in I$ such that $|x_i - L| < \varepsilon$ for all $i \geq i_0$.
Using the similar argument as in the sequence case, a limit of (x_i) is unique if it exists and we write $\lim_i x_i$ for its limits.

Example 2.5. **Appendix**: Using the notation given as before, let

$$I := \{P : \ P \text{ is a partitation on } [a,b] \ \}.$$

We say that $P_1 \leq P_2$ for $P_1, P_2 \in I$ if $P_1 \subseteq P_2$. Clearly, I is a directed set with this order. If we put $u_P := U((f,P)$, then we have

$$\lim_P u_P = \overline{\int_a^b} f.$$

In fact, let $\varepsilon > 0$. Then by the definition of an upper integral, there is $P_0 \in I$ such that

$$\overline{\int_a^b} f \leq U(f,P_0) \leq \overline{\int_a^b} f + \varepsilon.$$

Lemma 2.2 tells us that whenever $P \in I$ with $P \geq P_0$, we have $U(f,P) \leq U(f,P_0)$. Thus we have $|u_P - \overline{\int_a^b} f| < \varepsilon$ whenever $P \geq P_0$ as desired.

Proposition 2.6. *Let f and g both are bounded functions on $[a,b]$. With the notation as above, we always have*

(i)

$$\underline{\int_a^b} f \leq \overline{\int_a^b} f.$$

(ii) $\underline{\int_a^b}(-f) = -\overline{\int_a^b} f$.

(iii)

$$\underline{\int_a^b} f + \underline{\int_a^b} g \leq \underline{\int_a^b}(f+g) \leq \overline{\int_a^b}(f+g) \leq \overline{\int_a^b} f + \overline{\int_a^b} g.$$

Proof. Part (i) follows from Lemma 2.2 at once.
Part (ii) is clearly obtained by $L(-f,P) = -U(f,P)$.
For proving the inequality $\underline{\int_a^b} f + \underline{\int_a^b} g \leq \underline{\int_a^b}(f+g) \leq$ first. It is clear that we have $L(f,P) + L(g,P) \leq L(f+g,P)$ for all partitions P on $[a,b]$. Now let P_1 and P_2 be any partition on $[a,b]$. Then by Lemma 2.2, we have

$$L(f,P_1) + L(g,P_2) \leq L(f,P_1 \cup P_2) + L(g,P_1 \cup P_2) \leq L(f+g,P_1 \cup P_2) \leq \underline{\int_a^b}(f+g).$$

So, we have

$$\underline{\int_a^b} f + \underline{\int_a^b} g \leq \underline{\int_a^b} (f+g). \tag{2.2}$$

As before, we consider $-f$ and $-g$ in the Inequality 2.2, we get $\overline{\int_a^b}(f+g) \leq \overline{\int_a^b} f + \overline{\int_a^b} g$ as desired. $\square$

The following example shows the strict inequality in Proposition 2.6 (*iii*) may hold in general.

Example 2.7. *Define a function $f, g : [0,1] \to \mathbb{R}$ by*

$$f(x) = \begin{cases} 1 & \textit{if } \; x \in [0,1] \cap \mathbb{Q}; \\ -1 & \textit{otherwise.} \end{cases}$$

and

$$g(x) = \begin{cases} -1 & \textit{if } \; x \in [0,1] \cap \mathbb{Q}; \\ 1 & \textit{otherwise.} \end{cases}$$

Then it is easy to see that $f + g \equiv 0$ and

$$\overline{\int_0^1} f = \overline{\int_0^1} g = 1 \quad \textit{and} \quad \underline{\int_0^1} f = \underline{\int_0^1} g = -1.$$

So, we have

$$-2 = \underline{\int_a^b} f + \underline{\int_a^b} g < \underline{\int_a^b} (f+g) = 0 = \overline{\int_a^b} (f+g) < \overline{\int_a^b} f + \overline{\int_a^b} g = 2.$$

We can now reaching the main definition in this chapter.

Definition 2.8. *Let f be a bounded function on $[a,b]$. We say that f is Riemann integrable over $[a,b]$ if $\overline{\int_b^a} f = \underline{\int_a^b} f$. In this case, we write $\int_a^b f$ for this common value and it is called the Riemann integral of f over $[a,b]$.*
Also, write $R[a,b]$ for the class of Riemann integrable functions on $[a,b]$.

Proposition 2.9. *With the notation as above, $R[a,b]$ is a vector space over $\mathbb{R}$ and the integral*

$$\int_a^b : f \in R[a,b] \mapsto \int_a^b f \in \mathbb{R}$$

defines a linear functional, that is, $\alpha f + \beta g \in R[a,b]$ and $\int_a^b (\alpha f + \beta g) = \alpha \int_a^b f + \beta \int_a^b g$ for all $f, g \in R[a,b]$ and $\alpha, \beta \in \mathbb{R}$.

Proof. Let $f, g \in R[a,b]$ and $\alpha, \beta \in \mathbb{R}$. Notice that if $\alpha \geq 0$, it is clear that $\overline{\int_a^b} \alpha f = \alpha \overline{\int_a^b} f = \alpha \int_a^b f = \alpha \underline{\int_a^b} f = \underline{\int_a^b} \alpha f$. Also, if $\alpha < 0$, we have $\overline{\int_a^b} \alpha f = \alpha \underline{\int_a^b} f = \alpha \int_a^b f = \alpha \overline{\int_a^b} f = \underline{\int_a^b} \alpha f$. Therefore, we have $\int_a^b \alpha f = \alpha \int_a^b f$ for all $\alpha \in \mathbb{R}$. For showing $f + g \in R[a,b]$ and $\int_a^b (f+g) = \int_a^b f + \int_a^b g$, these will follows from Proposition 2.6 (*iii*) at once. The proof is finished. $\square$

The following result is the important characterization of a Riemann integrable function. Before showing this, we will use the following notation in the rest of this chapter.
For a partition $P : a = x_0 < x_1 < \cdots < x_n = b$ and $1 \le i \le n$, put

$$\omega_i(f, P) := \sup\{|f(x) - f(x')| : x, x' \in [x_{i-1}, x_i]\}.$$

It is easy to see that $U(f,P) - L(f,P) = \sum_{i=1}^{n} \omega_i(f,P)\Delta x_i$.

Theorem 2.10. *Let f be a bounded function on $[a,b]$. Then $f \in R[a,b]$ if and only if for all $\varepsilon > 0$, there is a partition $P : a = x_0 < \cdots < x_n = b$ on $[a,b]$ such that*

$$0 \le U(f,P) - L(f,P) = \sum_{i=1}^{n} \omega_i(f,P)\Delta x_i < \varepsilon. \tag{2.3}$$

Proof. Suppose that $f \in R[a,b]$. Let $\varepsilon > 0$. Then by the definition of the upper integral and lower integral of f, we can find the partitions P and Q such that $U(f,P) < \overline{\int_a^b} f + \varepsilon$ and $\underline{\int_a^b} f - \varepsilon < L(f,Q)$. By considering the partition $P \cup Q$, we see that

$$\underline{\int_a^b} f - \varepsilon < L(f,Q) \le L(f, P \cup Q) \le U(f, P \cup Q) \le U(f,P) < \overline{\int_a^b} f + \varepsilon.$$

Since $\int_a^b f = \overline{\int_a^b} f = \underline{\int_a^b} f$, we have $0 \le U(f, P\cup Q) - L(f, P \cup Q) < 2\varepsilon$. So, the partition $P \cup Q$ is as desired.
Conversely, let $\varepsilon > 0$, assume that the Inequality 2.3 above holds for some partition P. Notice that we have

$$L(f,P) \le \underline{\int_a^b} f \le \overline{\int_a^b} f \le U(f,P).$$

So, we have $0 \le \overline{\int_a^b} f - \underline{\int_a^b} f < \varepsilon$ for all $\varepsilon > 0$. The proof is finished. □

Remark 2.11. *Theorem 2.10 tells us that a bounded function f is Riemann integrable over $[a,b]$ if and only if the* "size" *of the discontinuous set of f is arbitrary small.*

Example 2.12. *Let $f : [0,1] \to \mathbb{R}$ be the function defined by*

$$f(x) = \begin{cases} \frac{1}{p} & \text{if } x = \frac{q}{p}, \text{ where } p, q \text{ are relatively prime positive integers;} \\ 0 & \text{otherwise.} \end{cases}$$

Then $f \in R[0,1]$.
(Notice that the set of all discontinuous points of f, say D, is just the set of all $(0,1] \cap \mathbb{Q}$. Since the set $(0,1] \cap \mathbb{Q}$ is countable, we can write $(0,1] \cap \mathbb{Q} = \{z_1, z_2, ...\}$. So, if we let $m(D)$ be the "size$''$ of the set D, then $m(D) = m(\bigcup_{i=1}^{\infty}\{z_i\}) = \sum_{i=1}^{\infty} m(\{z_i\}) = 0$, in here, you may think that the size of each set $\{z_i\}$ is 0.)

Proof. Let $\varepsilon > 0$. By Theorem 2.10, it aims to find a partition P on $[0,1]$ such that

$$U(f,P) - L(f,P) < \varepsilon.$$

Notice that for $x \in [0,1]$ such that $f(x) \ge \varepsilon$ if and only if $x = q/p$ for a pair of relatively prime positive integers p, q with $\frac{1}{p} \ge \varepsilon$. Since $1 \le q \le p$, there are only finitely many pairs of relatively prime positive integers p and q such that $f(\frac{q}{p}) \ge \varepsilon$. So, if we let $S := \{x \in [0,1] : f(x) \ge \varepsilon\}$, then S is a finite subset of $[0,1]$. Let L be the number of the elements in S. Then, for any partition $P : a = x_0 < \cdots < x_n = 1$, we have

$$\sum_{i=1}^{n} \omega_i(f,P)\Delta x_i = \Big(\sum_{i:[x_{i-1},x_i]\cap S=\emptyset} + \sum_{i:[x_{i-1},x_i]\cap S\neq\emptyset} \Big)\, \omega_i(f,P)\Delta x_i.$$

Notice that if $[x_{i-1}, x_i] \cap S = \emptyset$, then we have $\omega_i(f,P) \le \varepsilon$ and thus,

$$\sum_{i:[x_{i-1},x_i]\cap S=\emptyset} \omega_i(f,P)\Delta x_i \le \varepsilon \sum_{i:[x_{i-1},x_i]\cap S=\emptyset} \Delta x_i \le \varepsilon(1-0).$$

On the other hand, since there are at most $2L$ sub-intervals $[x_{i-1}, x_i]$ such that $[x_{i-1}, x_i] \cap S \neq \emptyset$ and $\omega_i(f,P) \le 1$ for all $i = 1, ..., n$, so, we have

$$\sum_{i:[x_{i-1},x_i]\cap S\neq\emptyset} \omega_i(f,P)\Delta x_i \le 1 \cdot \sum_{i:[x_{i-1},x_i]\cap S\neq\emptyset} \Delta x_i \le 2L\|P\|.$$

We can now conclude that for any partition P, we have

$$\sum_{i=1}^{n} \omega_i(f,P)\Delta x_i \le \varepsilon + 2L\|P\|.$$

So, if we take a partition P with $\|P\| < \varepsilon/(2L)$, then we have $\sum_{i=1}^{n} \omega_i(f,P)\Delta x_i \le 2\varepsilon$.
The proof is finished. □

Proposition 2.13. *Let f be a function defined on $[a,b]$. If f is either monotone or continuous on $[a,b]$, then $f \in R[a,b]$.*

Proof. We first show the case of f being monotone. We may assume that f is monotone increasing. Notice that for any partition $P : a = x_0 < \cdots < x_n = b$, we have $\omega_i(f,P) = f(x_i) - f(x_{i-1})$. So, if $\|P\| < \varepsilon$, we have

$$\sum_{i=1}^{n} \omega_i(f,P)\Delta x_i = \sum_{i=1}^{n} (f(x_i)-f(x_{i-1}))\Delta x_i < \|P\| \sum_{i=1}^{n} (f(x_i)-f(x_{i-1})) = \|P\|(f(b)-f(a)) < \varepsilon(f(b)-f(a)).$$

Therefore, $f \in R[a,b]$ if f is monotone.
Suppose that f is continuous on $[a,b]$. Then f is uniform continuous on $[a,b]$. Then for any $\varepsilon > 0$, there is $\delta > 0$ such that $|f(x) - f(x')| < \varepsilon$ as $x, x' \in [a,b]$ with $|x - x'| < \delta$. So, if we choose a partition P with $\|P\| < \delta$, then $\omega_i(f,P) < \varepsilon$ for all i. This implies that

$$\sum_{i=1}^{n} \omega_i(f,P)\Delta x_i \le \varepsilon \sum_{i=1}^{n} \Delta x_i = \varepsilon(b-a).$$

The proof is complete. □

Proposition 2.14. *We have the following assertions.*

(i) If $f, g \in R[a,b]$ with $f \le g$, then $\int_a^b f \le \int_a^b g$.

(ii) If $f \in R[a,b]$, then the absolute valued function $|f| \in R[a,b]$. In this case, we have $|\int_a^b f| \le \int_a^b |f|$.

Proof. For Part (i), it is clear that we have the inequality $U(f,P) \le U(g,P)$ for any partition P. So, we have $\int_a^b f = \overline{\int_a^b} f \le \overline{\int_a^b} g = \int_a^b g$.
For Part (ii), the integrability of $|f|$ follows immediately from Theorem 2.10 and the simple inequality $||f|(x') - |f|(x'')| \le |f(x') - f(x'')|$ for all $x', x'' \in [a,b]$. Thus, we have $U(|f|,P) - L(|f|,P) \le$

$U(f,P) - L(f,P)$ for any partition P on $[a,b]$.
Finally, since we have $-f \leq |f| \leq f$, by Part (i), we have $|\int_a^b f| \leq \int_a^b |f|$ at once. □

Proposition 2.15. *Let $a < c < b$. We have $f \in R[a,b]$ if and only if the restrictions $f|_{[a,c]} \in R[a,c]$ and $f|_{[c,b]} \in R[c,b]$. In this case we have*

$$\int_a^b f = \int_a^c f + \int_c^b f. \tag{2.4}$$

Proof. Let $f_1 := f|_{[a,c]}$ and $f_2 := f|_{[c,b]}$.
It is clear that we always have

$$U(f_1,P_1) - L(f_1,P_1) + U(f_2,P_2) - L(f_2,P_2) = U(P,f) - L(f,P)$$

for any partition P_1 on $[a,c]$ and P_2 on $[c,b]$ with $P = P_1 \cup P_2$.
From this, we can show the sufficient condition at once.
For showing the necessary condition, since $f \in R[a,b]$, for any $\varepsilon > 0$, there is a partition Q on $[a,b]$ such that $U(f,Q) - L(f,Q) < \varepsilon$ by Theorem 2.10. Notice that there are partitions P_1 and P_2 on $[a,c]$ and $[c,b]$ respectively such that $P := Q \cup \{c\} = P_1 \cup P_2$. Thus, we have

$$U(f_1,P_1) - L(f_1,P_1) + U(f_2,P_2) - L(f_2,P_2) = U(f,P) - L(f,P) \leq U(f,Q) - L(f,Q) < \varepsilon.$$

So, we have $f_1 \in R[a,c]$ and $f_2 \in R[c,b]$.
It remains to show the Equation 2.4 above. Notice that for any partition P_1 on $[a,c]$ and P_2 on $[c,b]$, we have

$$L(f_1,P_1) + L(f_2,P_2) = L(f, P_1 \cup P_2) \leq \underline{\int_a^b} f = \int_a^b f.$$

So, we have $\int_a^c f + \int_c^b f \leq \int_a^b f$. Then the inverse inequality can be obtained at once by considering the function $-f$. Then the resulted is obtained by using Theorem 2.10. □

Proposition 2.16. *Let f and g be Riemann integrable functions defined ion $[a,b]$. Then the pointwise product function $f \cdot g \in R[a,b]$.*

Proof. We first show that the square function f^2 is Riemann integrable. In fact, if we let $M = \sup\{|f(x)| : x \in [a,b]\}$, then we have $\omega_k(f^2,P) \leq 2M\omega_k(f,P)$ for any partition $P : a = x_0 < \cdots < a_n = b$ because we always have $|f^2(x) - f^2(x')| \leq 2M|f(x) - f(x')|$ for all $x, x' \in [a,b]$. Then by Theorem 2.10, the square function $f^2 \in R[a,b]$.
This, together with the identity $f \cdot g = \frac{1}{2}((f+g)^2 - f^2 - g^2)$. The result follows. □

Remark 2.17. *In the proof of Proposition 2.16, we have shown that if $f \in R[a,b]$, then so is its square function f^2. However, the converse does not hold. For example, if we consider $f(x) = 1$ for $x \in \mathbb{Q} \cap [0,1]$ and $f(x) = -1$ for $x \in \mathbb{Q}^c \cap [0,1]$, then $f \notin R[0,1]$ but $f^2 \equiv 1$ on $[0,1]$.*

Proposition 2.18. (Mean Value Theorem for Integrals)
Let f and g be the functions defined on $[a,b]$. Assume that f is continuous and g is a non-negative Riemann integrable function. Then, there is a point $\xi \in (a,b)$ such that

$$\int_a^b f(x)g(x)dx = f(\xi)\int_a^b g(x)dx. \tag{2.5}$$

Proof. By the continuity of f on $[a,b]$, there exist two points x_1 and x_2 in $[a,b]$ such that

$$f(x_1) = m := \min f(x); \text{ and } f(x_2) = M := \max f(x).$$

We may assume that $a \leq x_1 < x_2 \leq b$. From this, since $g \leq 0$, we have

$$mg(x) \leq f(x)g(x) \leq Mg(x)$$

for all $x \in [a, b]$. From this and Proposition 2.16 above, we have

$$m \int_a^b g \leq \int_a^b fg \leq M \int_a^b g.$$

So, if $\int_a^b g = 0$, then the result follows at once.
We may now suppose that $\int_a^b g > 0$. The above inequality shows that

$$m = f(x_1) \leq \frac{\int_a^b fg}{\int_a^b g} \leq f(x_2) = M.$$

Therefore, there is a point $\xi \in [x_1, x_2] \subseteq [a, b]$ so that the Equation 2.5 holds by using the Intermediate Value Theorem for the function f. Thus, it remains to show that such element ξ can be chosen in (a, b).
Let $a \leq x_1 < x_2 \leq b$ be as above.
If x_1 and x_2 can be found so that $a < x_1 < x_2 < b$, then the result is proved immediately since $\xi \in [x_1, x_2] \subset (a, b)$ in this case.
Now suppose that x_1 or x_2 does not exist in (a, b), i.e., $m = f(a) < f(x)$ for all $x \in (a, b]$ or $f(x) < f(b) = M$ for all $x \in [a, b)$.
Claim 1: If $f(a) < f(x)$ for all $x \in (a, b]$, then $\int_a^b fg > f(a) \int_a^b g$ and hence, $\xi \in (a, x_2] \subseteq (a, b]$.
For showing **Claim1**, put $h(x) := f(x) - f(a)$ for $x \in [a, b]$. Then h is continuous on $[a, b]$ and $h > 0$ on $(a, b]$. This implies that $\int_c^d h > 0$ for any subinterval $[c, d] \subseteq [a, b]$. (**Why**?)
On the other hand, since $\underline{\int}_a^b g = \int_a^b g > 0$, there is a partition $P : a = x_0 < \cdots < x_n = b$ so that $L(g, P) > 0$. This implies that $m_k(g, P) > 0$ for some sub-interval $[x_{k-1}, x_k]$. Therefore, we have

$$\int_a^b hg \geq \int_{x_{k-1}}^{x_k} hg \geq m_k(g, P) \int_{x_{k-1}}^{x_k} h > 0.$$

Hence, we have $\int_a^b fg > f(a) \int_a^b g$. **Claim 1** follows.
Similarly, one can show that if $f(x) < f(b) = M$ for all $x \in [a, b)$, then we have $\int_a^b fg < f(b) \int_a^b g$. This, together with **Claim 1** give us that such ξ can be found in (a, b). The proof is finished. □

Now if $f \in R[a, b]$, then by Proposition 2.15, we can define a function $F : [a, b] \to \mathbb{R}$ by

$$F(c) = \begin{cases} 0 & \text{if } c = a \\ \int_a^c f & \text{if } a < c \leq b. \end{cases} \tag{2.6}$$

Theorem 2.19. Fundamental Theorem of Calculus: *With the notation as above, assume that $f \in R[a, b]$, we have the following assertion.*

*(i) If there is a continuous function F on $[a, b]$ which is differentiable on (a, b) with $F' = f$, then $\int_a^b f = F(b) - F(a)$. In this case, F is called an indefinite integral of f. (**note**: if F_1 and F_2 both are the indefinite integrals of f, then by the Mean Value Theorem, we have $F_2 = F_1 +$ constant).*

(ii) The function F defined as in Eq. 2.6 above is continuous on $[a, b]$. Furthermore, if f is continuous on $[a, b]$, then F' exists on (a, b) and $F' = f$ on (a, b).

Proof. For Part (i), notice that for any partition $P : a = x_0 < \cdots < x_n = b$, then by the Mean Value Theorem, for each $[x_{i-1}, x_i]$, there is $\xi_i \in (x_{i-1}, x_i)$ such that $F(x_i) - F(x_{i-1}) = F'(\xi_i)\Delta x_i = f(\xi_i)\Delta x_i$. So, we have

$$L(f, P) \le \sum f(\xi_i)\Delta x_i = \sum F(x_i) - F(x_{i-1}) = F(b) - F(a) \le U(f, P)$$

for all partitions P on $[a, b]$. This gives

$$\int_a^b f = \underline{\int_a^b} f \le F(b) - F(a) \le \overline{\int_a^b} f = \int_a^b f$$

as desired.

For showing the continuity of F in Part (ii), let $a < c < x < b$. If $|f| \le M$ on $[a, b]$, then we have $|F(x) - F(c)| = |\int_c^x f| \le M(x - c)$. So, $\lim_{x\to c+} F(x) = F(c)$. Similarly, we also have $\lim_{x\to c-} F(x) = F(c)$. Thus F is continuous on $[a, b]$.

Now assume that f is continuous on $[a, b]$. Notice that for any $t > 0$ with $a < c < c + t < b$, we have

$$\inf_{x\in[c,c+t]} f(x) \le \frac{1}{t}(F(c + t) - F(c)) = \frac{1}{t}\int_c^{c+t} f \le \sup_{x\in[c,c+t]} f(x).$$

Since f is continuous at c, we see that $\lim\limits_{t\to 0+} \frac{1}{t}(F(c+t)-F(c)) = f(c)$. Similarly, we have $\lim\limits_{t\to 0-} \frac{1}{t}(F(c+t) - F(c)) = f(c)$. So, we have $F'(c) = f(c)$ as desired. The proof is finished. □

Definition 2.20. *For each function f on $[a, b]$ and a partition $P : a = x_0 < \cdots < x_n = b$, we call $R(f, P, \{\xi_i\}) := \sum_{I=1}^{N} f(\xi_i)\Delta x_i$, where $\xi_i \in [x_{i-1}, x_i]$, the Riemann sum of f over $[a, b]$.*
We say that the Riemann sum $R(f, P, \{\xi_i\})$ converges to a number A as $\|P\| \to 0$, write $A = \lim\limits_{\|P\|\to 0} R(f, P, \{\xi_i\})$, if for any $\varepsilon > 0$, there is $\delta > 0$ such that

$$|A - R(f, P, \{\xi_i\})| < \varepsilon$$

whenever $\|P\| < \delta$ and for any $\xi_i \in [x_{i-1}, x_i]$.

Proposition 2.21. *Let f be a function defined on $[a, b]$. If the limit $\lim\limits_{\|P\|\to 0} R(f, P, \{\xi_i\}) = A$ exists, then f is automatically bounded.*

Proof. Suppose that f is unbounded. Then by the assumption, there exists a partition $P : a = x_0 < \cdots < x_n = b$ such that $|\sum_{k=1}^{n} f(\xi_k)\Delta x_k| < 1 + |A|$ for any $\xi_k \in [x_{k-1}, x_k]$. Since f is unbounded, we may assume that f is unbounded on $[a, x_1]$. In particular, we choose $\xi_k = x_k$ for $k = 2, ..., n$. Also, we can choose $\xi_1 \in [a, x_1]$ such that

$$|f(\xi_1)|\Delta x_1 > 1 + |A| + |\sum_{k=2}^{n} f(x_k)\Delta x_k|.$$

It leads to a contradiction because we have $1 + |A| > |f(\xi_1)|\Delta x_1 - |\sum_{k=2}^{n} f(x_k)\Delta x_k|$. The proof is finished. □

Lemma 2.22. *$f \in R[a, b]$ if and only if for any $\varepsilon > 0$, there is $\delta > 0$ such that $U(f, P) - L(f, P) < \varepsilon$ whenever $\|P\| < \delta$.*

Proof. The converse follows from Theorem 2.10.

Assume that f is integrable over $[a, b]$. Let $\varepsilon > 0$. Then there is a partition $Q : a = y_0 < ... < y_l = b$ on

$[a,b]$ such that $U(f,Q)-L(f,Q)<\varepsilon$. Now take $0<\delta<\varepsilon/l$. Suppose that $P: a=x_0<...<x_n=b$ with $\|P\|<\delta$. Then we have

$$U(f,P)-L(f,P)=I+II$$

where

$$I=\sum_{i:Q\cap[x_{i-1},x_i]=\emptyset}\omega_i(f,P)\Delta x_i;$$

and

$$II=\sum_{i:Q\cap[x_{i-1},x_i]\neq\emptyset}\omega_i(f,P)\Delta x_i$$

Notice that we have

$$I\leq U(f,Q)-L(f,Q)<\varepsilon$$

and

$$II\leq (M-m)\sum_{i:Q\cap[x_{i-1},x_i]\neq\emptyset}\Delta x_i\leq (M-m)\cdot 2l\cdot\frac{\varepsilon}{l}=2(M-m)\varepsilon.$$

The proof is finished. □

Theorem 2.23. *$f\in R[a,b]$ if and only if the Riemann sum $R(f,P,\{\xi_i\})$ is convergent. In this case, $R(f,P,\{\xi_i\})$ converges to $\int_a^b f(x)dx$ as $\|P\|\to 0$.*

Proof. For the proof $(\Rightarrow)$: we first note that we always have

$$L(f,P)\leq R(f,P,\{\xi_i\})\leq U(f,P)$$

and

$$L(f,P)\leq\int_a^b f(x)dx\leq U(f,P)$$

for any partition P and $\xi_i\in[x_{i-1},x_i]$.

Now let $\varepsilon>0$. Lemma 2.22 gives $\delta>0$ such that $U(f,P)-L(f,P)<\varepsilon$ as $\|P\|<\delta$. Then we have

$$|\int_a^b f(x)dx-R(f,P,\{\xi_i\})|<\varepsilon$$

as $\|P\|<\delta$ and $\xi_i\in[x_{i-1},x_i]$. The necessary part is proved and $R(f,P,\{\xi_i\})$ converges to $\int_a^b f(x)dx$.
For $(\Leftarrow)$: assume that there is a number A such that for any $\varepsilon>0$, there is $\delta>0$, we have

$$A-\varepsilon<R(f,P,\{\xi_i\})<A+\varepsilon$$

for any partition P with $\|P\|<\delta$ and $\xi_i\in[x_{i-1},x_i]$.
Note that f is automatically bounded in this case by Proposition 2.21.
Now fix a partition P with $\|P\|<\delta$. Then for each $[x_{i-1},x_i]$, choose $\xi_i\in[x_{i-1},x_i]$ such that $M_i(f,P)-\varepsilon\leq f(\xi_i)$. This implies that we have

$$U(f,P)-\varepsilon(b-a)\leq R(f,P,\{\xi_i\})<A+\varepsilon.$$

Thus, we have shown that for any $\varepsilon>0$, there is a partition $\mathcal{P}$ such that

$$\overline{\int_a^b} f(x)dx\leq U(f,P)\leq A+\varepsilon(1+b-a). \tag{2.7}$$

By considering $-f$, note that the Riemann sum of $-f$ will converge to $-A$. The inequality 2.7 will imply that for any $\varepsilon > 0$, there is a partition P such that

$$A - \varepsilon(1 + b - a) \leq \underline{\int_a^b} f(x)dx \leq \overline{\int_a^b} f(x)dx \leq A + \varepsilon(1 + b - a).$$

The proof is complete. □

Theorem 2.24. *Let $f \in R[c,d]$ and let $\phi : [a,b] \longrightarrow [c,d]$ be a strictly increasing C^1 function with $f(a) = c$ and $f(b) = d$.*
Then $f \circ \phi \in R[a,b]$, moreover, we have

$$\int_c^d f(x)dx = \int_a^b f(\phi(t))\phi'(t)dt.$$

Proof. Let $A = \int_c^d f(x)dx$. By using Theorem 2.23, we need to show that for all $\varepsilon > 0$, there is $\delta > 0$ such that

$$|A - \sum f(\phi(\xi_k))\phi'(\xi_k)\triangle t_k| < \varepsilon$$

for all $\xi_k \in [t_{k-1}, t_k]$ whenever $Q : a = t_0 < ... < t_m = b$ with $\|Q\| < \delta$.
Now let $\varepsilon > 0$. Then by Lemma 2.22 and Theorem 2.23, there is $\delta_1 > 0$ such that

$$|A - \sum f(\eta_k)\triangle x_k| < \varepsilon \tag{2.8}$$

and

$$\sum \omega_k(f, P)\triangle x_k < \varepsilon \tag{2.9}$$

for all $\eta_k \in [x_{k-1}, x_k]$ whenever $P : c = x_0 < ... < x_m = d$ with $\|P\| < \delta_1$.
Now put $x = \phi(t)$ for $t \in [a,b]$.
Now since ϕ and ϕ' are continuous on $[a,b]$, there is $\delta > 0$ such that $|\phi(t) - \phi(t')| < \delta_1$ and $|\phi'(t) - \phi'(t')| < \varepsilon$ for all $t, t'\ in[a,b]$ with $|t - t'| < \delta$.
Now let $Q : a = t_0 < ... < t_m = b$ with $\|Q\| < \delta$. If we put $x_k = \phi(t_k)$, then $P : c = x_0 < .... < x_m = d$ is a partition on $[c,d]$ with $\|P\| < \delta_1$ because ϕ is strictly increasing.
Note that the Mean Value Theorem implies that for each $[t_{k-1}, t_k]$, there is $\xi_k^* \in (t_{k-1}, t_k)$ such that

$$\triangle x_k = \phi(t_k) - \phi(t_{k-1}) = \phi'(\xi_k^*)\Delta t_k.$$

This yields that

$$|\triangle x_k - \phi'(\xi_k)\triangle t_k| < \varepsilon\Delta t_k \tag{2.10}$$

for any $\xi_k \in [t_{k-1}, t_k]$ for all $k = 1, ..., m$ because of the choice of δ.
Now for any $\xi_k \in [t_{k-1}, t_k]$, we have

$$\begin{aligned} |A - \sum f(\phi(\xi_k))\phi'(\xi_k)\triangle t_k| \leq\ & |A - \sum f(\phi(\xi_k^*))\phi'(\xi_k^*)\triangle t_k| \\ & + |\sum f(\phi(\xi_k^*))\phi'(\xi_k^*)\triangle t_k - \sum f(\phi(\xi_k^*))\phi'(\xi_k)\triangle t_k| \\ & + |\sum f(\phi(\xi_k^*))\phi'(\xi_k)\triangle t_k - \sum f(\phi(\xi_k))\phi'(\xi_k)\triangle t_k| \end{aligned} \tag{2.11}$$

Notice that inequality 2.8 implies that

$$|A - \sum f(\phi(\xi_k^*))\phi'(\xi_k^*)\triangle t_k| = |A - \sum f(\phi(\xi_k^*))\triangle x_k| < \varepsilon.$$

Moreover, since we have $|\phi'(\xi_k^*) - \phi'(\xi_k)| < \varepsilon$ for all $k = 1, .., m$, we have

$$|\sum f(\phi(\xi_k^*))\phi'(\xi_k^*)\triangle t_k - \sum f(\phi(\xi_k^*))\phi'(\xi_k)\triangle t_k| \leq M(b-a)\varepsilon$$

where $|f(x)| \leq M$ for all $x \in [c,d]$.
On the other hand, by using inequality 2.10 we have

$$|\phi'(\xi_k)\triangle t_k| \leq \triangle x_k + \varepsilon \triangle t_k$$

for all k. This, together with inequality 2.9 imply that

$$\begin{aligned}
&|\sum f(\phi(\xi_k^*))\phi'(\xi_k)\triangle t_k - \sum f(\phi(\xi_k))\phi'(\xi_k)\triangle t_k| \\
&\leq \sum \omega_k(f,P)|\phi'(\xi_k)\triangle t_k| \ (\because \phi(\xi_k^*), \phi(\xi_k) \in [x_{k-1}, x_k]) \\
&\leq \sum \omega_k(f,P)(\triangle x_k + \varepsilon \triangle t_k) \\
&\leq \varepsilon + 2M(b-a)\varepsilon.
\end{aligned}$$

Finally by inequality 2.11, we have

$$|A - \sum f(\phi(\xi_k))\phi'(\xi_k)\triangle t_k| \leq \varepsilon + M(b-a)\varepsilon + \varepsilon + 2M(b-a)\varepsilon.$$

The proof is complete. □

3. Improper Riemann Integrals

Definition 3.1. *Let $-\infty < a < b < \infty$.*

(i) Let f be a function defined on $[a,\infty)$. Assume that the restriction $f|_{[a,T]}$ is integrable over $[a,T]$ for all $T > a$. Put $\displaystyle\int_a^\infty f := \lim_{T\to\infty} \int_a^T f$ if this limit exists.
Similarly, we can define $\int_{-\infty}^b f$ if f is defined on $(-\infty, b]$.

(ii) If f is defined on $(a,b]$ and $f|_{[c,b]} \in R[c,b]$ for all $a < c < b$. Put $\displaystyle\int_a^b f := \lim_{c\to a+} \int_c^b f$ if it exists.
Similarly, we can define $\int_a^b f$ if f is defined on $[a,b)$.

(iii) As f is defined on $\mathbb{R}$, if $\int_0^\infty f$ and $\int_{-\infty}^0 f$ both exist, then we put $\int_{-\infty}^\infty f = \int_{-\infty}^0 f + \int_0^\infty f$.

In the cases above, we call the resulting limits the improper Riemann integrals of f and say that the integrals are convergent.

Example 3.2. *Define (*formally*) an improper integral $\Gamma(s)$ (called the Γ-function) as follows:*

$$\Gamma(s) := \int_0^\infty x^{s-1}e^{-x}dx$$

for $s \in \mathbb{R}$. Then $\Gamma(s)$ is convergent if and only if $s > 0$.

Proof. Put $I(s) := \int_0^1 x^{s-1}e^{-x}dx$ and $II(s) := \int_1^\infty x^{s-1}e^{-x}dx$. We first claim that the integral $II(s)$ is convergent for all $s \in \mathbb{R}$.
In fact, if we fix $s \in \mathbb{R}$, then we have

$$\lim_{x\to\infty} \frac{x^{s-1}}{e^{x/2}} = 0.$$

So there is $M > 1$ such that $\frac{x^{s-1}}{e^{x/2}} \leq 1$ for all $x \geq M$. Thus we have

$$0 \leq \int_M^\infty x^{s-1}e^{-x}dx \leq \int_M^\infty e^{-x/2}dx < \infty.$$

Therefore we need to show that the integral $I(s)$ is convergent if and only if $s > 0$.
Note that for $0 < \eta < 1$, we have

$$0 \le \int_\eta^1 x^{s-1}e^{-x}dx \le \int_\eta^1 x^{s-1}dx = \begin{cases} \frac{1}{s}(1-\eta^s) & \text{if } s-1 \ne -1; \\ -\ln\eta & \text{otherwise .} \end{cases}$$

Thus the integral $I(s) = \lim_{\eta\to 0+} \int_\eta^1 x^{s-1}e^{-x}dx$ is convergent if $s > 0$.
Conversely, we also have

$$\int_\eta^1 x^{s-1}e^{-x}dx \ge e^{-1}\int_\eta^1 x^{s-1}dx = \begin{cases} \frac{e^{-1}}{s}(1-\eta^s) & \text{if } s-1 \ne -1; \\ -e^{-1}\ln\eta & \text{otherwise .} \end{cases}$$

So if $s \le 0$, then $\int_\eta^1 x^{s-1}e^{-x}dx$ is divergent as $\eta \to 0+$. The result follows. □